

INTEGRALI I TEORIJA POLJA

- zadaci za vežbu -

1. Izračunati direktno krivolinijski integral:

$$I = \oint_C ydx + x^2dy + zdz$$

duž presečne krive površi:

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{i} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b},$$

orjentisane u pozitivnom smeru ako se posmatra sa z ose za vrednosti $z > 2$ ($a \geq b > 0$).

Rešenje. Presek cilindrične površi $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b}$ i ravni $z = 0$ je elipsa:

$$\frac{(x - \frac{a}{2})^2}{a^2/2} + \frac{(y - \frac{b}{2})^2}{b^2/2} = 1,$$

čije su parametarske jednačine:

$$x = \frac{a}{2} + \frac{a}{\sqrt{2}} \cos t, \quad y = \frac{b}{2} + \frac{b}{\sqrt{2}} \sin t,$$

za $t \in [0, 2\pi]$. Dalje, na osnovu: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ nalazimo:

$$z = 1 + \frac{1}{\sqrt{2}} \cos t + \frac{1}{\sqrt{2}} \sin t,$$

za $t \in [0, 2\pi]$. Tako dobijena parametarizacija krive C je pozitivna ako se posmatra kriva sa z ose za vrednosti $z > 2$. Na osnovu parametarskih jednačina krive C :

$$x = x(t) = \frac{a}{2} + \frac{a}{\sqrt{2}} \cos t,$$

$$y = y(t) = \frac{b}{2} + \frac{b}{\sqrt{2}} \sin t,$$

$$z = z(t) = 1 + \frac{1}{\sqrt{2}} \cos t + \frac{1}{\sqrt{2}} \sin t,$$

za $t \in [0, 2\pi]$ nalazimo odgovarajuće diferencijale:

$$dx = -\frac{a}{\sqrt{2}} \sin t dt,$$

$$dy = \frac{b}{\sqrt{2}} \cos t dt,$$

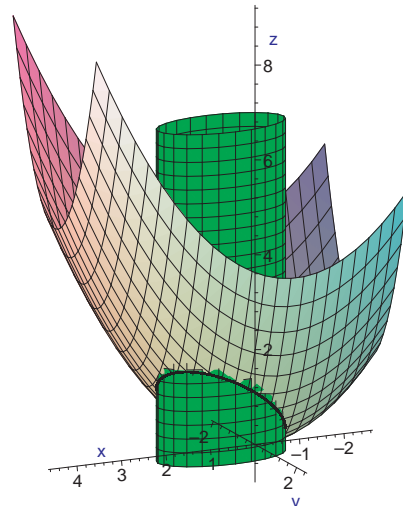
$$dz = \frac{1}{\sqrt{2}}(-\sin t + \cos t) dt.$$

Sveukupno, na osnovu prethodne parametarizacije:

$$\begin{aligned}
 I &= \int_0^{2\pi} \left(\left(\frac{b}{2} + \frac{b}{\sqrt{2}} \sin t \right) \left(-\frac{a}{\sqrt{2}} \sin t \right) + \left(\frac{a}{2} + \frac{a}{\sqrt{2}} \cos t \right)^2 \left(\frac{b}{\sqrt{2}} \cos t \right) \right. \\
 &+ \left. \left(1 + \frac{1}{\sqrt{2}} \cos t + \frac{1}{\sqrt{2}} \sin t \right) \left(\frac{1}{\sqrt{2}} (-\sin t + \cos t) \right) \right) dt \\
 &= \int_0^{2\pi} \left(\frac{a^2 b}{2\sqrt{2}} \cos^3 t + \frac{ba^2 + ab + 2}{2} \cos^2 t + \frac{a^2 b + 4}{4\sqrt{2}} \cos t - \frac{ab + 2}{2\sqrt{2}} \sin t - \frac{ab + 1}{2} \right) dt \\
 &= \frac{ba^2 + ab + 2}{2} \cdot \int_0^{2\pi} \cos^2 t dt - \frac{ab + 1}{2} \cdot \int_0^{2\pi} dt = \frac{ba^2 + ab + 2}{2} \cdot \pi - \frac{ab + 1}{2} \cdot 2\pi \\
 &= \frac{ab(a - 1)\pi}{2}.
 \end{aligned}$$

Proverimo prethodni rezultat primenom STOKESOVE formule. Kontura C dobija se u preseku parabolida: $z = x^2/a^2 + y^2/b^2$ i cilindra: $x^2/a^2 + y^2/b^2 = x/a + y/b$. Primetimo da su funkcije $P = P(x, y, z) = y$, $Q = Q(x, y, z) = x^2$ i $R = R(x, y, z) = z$ sa neprekidnim parcijalnim izvodima na površi paraboloida. Neka je S unutrašnja strana paraboloida koja se oslanja na orjentisanu konturu C . Samim tim, prema STOKESOVJ formuli:

$$\begin{aligned}
 I &= \oint_L y dx + x^2 dy + z dz \\
 &= \iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x^2 & z \end{vmatrix} \\
 &= \iint_S (2x - 1) dxdy.
 \end{aligned}$$



Projekcija površi S na Oxy ravan je data skupom: $D_{x,y}: \frac{(x - a/2)^2}{a^2/2} + \frac{(x - b/2)^2}{b^2/2} \leq 1$.

Samim tim:

$$I = \iint_{D_{x,y}} (2x - 1) dxdy.$$

Uvedimo polarne koordinate: $x = a/2 + a/\sqrt{2}\rho \cos \varphi \wedge y = b/2 + b/\sqrt{2}\rho \sin \varphi$ za $(\rho, \varphi) \in D' = \{(\rho, \varphi) \mid 0 \leq \rho \leq 1 \wedge 0 \leq \varphi \leq 2\pi\}$. Tada je $|J| = \frac{ab}{2}\rho$. Samim tim prethodni integral, posle uvođenja novih promenljivih, ima vrednost:

$$\begin{aligned}
 I &= \iint_{D'} (a + a\sqrt{2}\rho \cos \varphi - 1) \frac{ab}{2} \rho d\rho d\varphi = \frac{ab}{2} \int_0^{2\pi} \left(\int_0^1 ((a - 1)\rho + a\sqrt{2}\rho^2 \cos \varphi) d\rho \right) d\varphi \\
 &= \frac{ab}{2} \int_0^{2\pi} \left(\frac{a - 1}{2} + \frac{a\sqrt{2}}{3} \cos \varphi \right) d\varphi = \frac{ab(a - 1)}{4} \int_0^{2\pi} d\varphi = \frac{ab\pi(a - 1)}{2}.
 \end{aligned}$$

2. Zadan je krivolinijski integral:

$$\oint_C ydx + zdy + xdz,$$

duž zatvorene presečne krive C sledećih površi: $x^2 + y^2 = r^2$ i $x^2 = rz$ ($r > 0$). Izračunati vrednost krivolinijskog integrala ukoliko je kriva C orjetisana pozitivno ako se posmatra sa z ose za $z > r$.

Rešenje. Koristeći se parametarskim jednačinama kružnice $x^2 + y^2 = r^2$ (u ravni Oxy) zaključujemo da presečna kriva C u prostoru ima sledeće parametarske jednačine:

$$x = x(\varphi) = r \cos \varphi \quad \wedge \quad y = y(\varphi) = r \sin \varphi \quad \wedge \quad z = z(\varphi) = r \cos^2 \varphi,$$

gde je $\varphi \in [0, 2\pi]$. Tako dobijena parametarizacija krive C je pozitivna ako se posmatra kriva sa z ose za vrednosti $z > r$. Sveukupno, na osnovu prethodne parametarizacije:

$$\begin{aligned} \oint_C ydx + zdy + xdz &= \int_0^{2\pi} (-r^2 \sin^2 \varphi + r^2 \cos^3 \varphi - 2r^2 \cos^2 \varphi \sin \varphi) d\varphi \\ &= -4r^2 \cdot \left(\int_0^{\pi/2} \sin^2 \varphi d\varphi \right) + r^2 \cdot \underbrace{\left(\int_0^{2\pi} \cos^3 \varphi d\varphi \right)}_{=0} \\ &\quad - 2r^2 \cdot \underbrace{\left(\int_0^{2\pi} \cos^2 \varphi \cdot \sin \varphi d\varphi \right)}_{=0} = -4r^2 \cdot \frac{1}{2} B\left(\frac{1}{2}, \frac{3}{2}\right) \\ &= -r^2 \pi. \end{aligned}$$

Primetimo da su funkcije $P = P(x, y, z) = y$, $Q = Q(x, y, z) = z$ i $R = R(x, y, z) = x$ sa neprekidnim parcijalnim izvodima na površi S koja se oslanja na orjentisanu konturu C . Samim tim, prema STOKESovoj formuli:

$$\oint_C ydx + zdy + xdz = \left| \begin{array}{ccc} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{array} \right| = - \iint_S dydz + dzdx + dxdy.$$

Formirajmo tri površinska integrala:

$$J_1 = - \iint_S dydz, \quad J_2 = - \iint_S dzdx, \quad J_3 = - \iint_S dxdy.$$

Računajući prethodne površinske integrale, preko odgovarajućih projekcija na koordinatne ravni, nalazimo $J_1 = 0$, $J_2 = 0$ i $J_3 = -r^2\pi$. Odatle dobijamo isti rezultat:

$$\oint_C ydx + zdy + xdz = J_1 + J_2 + J_3 = -r^2\pi.$$

3. Izračunati krivolinijski integral:

$$I = \oint_L x^2 y^3 dx + dy + z dz$$

gde je L pozitivno orjetisana kružnica data sa: $x^2 + y^2 = r^2$ i $z = 0$, ukoliko se posmata sa pozitivnog dela z -ose ($r > 0$).

Rešenje. Parametarske jednačine kružnice L u ravni $z = 0$ glase:

$$x = x(t) = r \cos t, \quad y = r \sin t, \quad z = 0,$$

gde je $t \in [0, 2\pi]$. Tako dobijena parametarizacija kružnice L je pozitivna ako se posmatra sa pozitivnog dela z -ose. Sveukupno, na osnovu prethodne parametarizacije:

$$\begin{aligned} I &= \oint_L x^2 y^3 dx + dy + z dz = \int_0^{2\pi} \left((r \cos t)^2 (r \sin t)^3 (-r \sin t dt) + (r \cos t dt) \right) \\ &= -r^6 \int_0^{2\pi} \cos^2 t \sin^4 t dt + r \int_0^{2\pi} \cos t dt = -4r^6 \int_0^{\pi/2} \cos^2 t \sin^4 t dt = -4r^6 \frac{1}{2} B\left(\frac{3}{2}, \frac{5}{2}\right) \\ &= -\frac{r^6 \pi}{8}. \end{aligned}$$

Proverimo prethodni rezultat STOKESovom formulom. Primitimo da su funkcije $P = P(x, y, z) = x^2 y^3$, $Q = Q(x, y, z) = 1$ i $R = R(x, y, z) = z$ sa neprekidnim parcijalnim izvodima na površi S koja se oslanja na orjentisanu konturu L . Samim tim, prema STOKESovoj formuli:

$$I = \oint_L x^2 y^3 dx + dy + z dz = \iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y^3 & 1 & z \end{vmatrix} = -3 \iint_S x^2 y^2 dx dy.$$

Primitimo da se u ovom zadatku površ S podudara sa Oxy -projekcijom $D = \{(x, y) | x^2 + y^2 \leq r^2\}$, odatle:

$$I = -3 \iint_D x^2 y^2 dx dy.$$

Uvedimo polarne koordinate: $x = \rho \cos \varphi \wedge y = \rho \sin \varphi$ za $(\rho, \varphi) \in D' = \{(\rho, \varphi) | 0 \leq \rho \leq r \wedge 0 \leq \varphi \leq 2\pi\}$. Tada je $|J| = \rho$. Samim tim prethodni integral, posle uvođenja novih promenljivih, ima vrednost:

$$\begin{aligned} I &= -3 \iint_{D'} (\rho \cos \varphi)^2 (\rho \sin \varphi)^2 \rho d\rho d\varphi = -3 \int_0^r \rho^5 d\rho \int_0^{2\pi} \sin^2 \varphi \cos^2 \varphi d\varphi \\ &= -3 \cdot \frac{r^6}{6} \cdot 4 \int_0^{\pi/2} \sin^2 \varphi \cos^2 \varphi d\varphi = -r^6 B\left(\frac{3}{2}, \frac{3}{2}\right) = -\frac{r^6 \pi}{8}. \end{aligned}$$

4. Izračunati integral:

$$I = \iint_D xy \, dx dy,$$

ako je oblast D ograničena pravama $2x - y = 1$, $2x - y = 3$, $x + y = -2$ i $x + y = 0$.

Rešenje. Uvedimo smenu promenljivih:

$$\begin{cases} u = u(x, y) = 2x - y, \\ v = v(x, y) = x + y. \end{cases}$$

Prethodnom smenom ostvaruje se uzajamno jednoznačno preslikavanje oblasti:

$$D = \{(x, y) \mid 2x - 3 \leq y \leq 2x - 1, -x - 2 \leq y \leq -x\}$$

i

$$D_1 = \{(u, v) \mid 1 \leq u \leq 3, -2 \leq v \leq 0\}.$$

Kako je $x = \frac{1}{3}(u + v)$ i $y = \frac{1}{3}(-u + 2v)$, tada nalazimo: $J = \frac{D(x, y)}{D(u, v)} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} = \frac{1}{3}$.

Samim tim, smenom promenljivih, nalazimo vrednost integrala:

$$\begin{aligned} I &= \iint_D xy \, dx dy = \iint_{D_1} \frac{1}{9}(u + v)(2v - u)|J| \, dudv \\ &= \frac{1}{27} \int_1^3 \left(\int_{-2}^0 (u + v)(2v - u) \, dv \right) du = -\frac{44}{81}. \end{aligned}$$

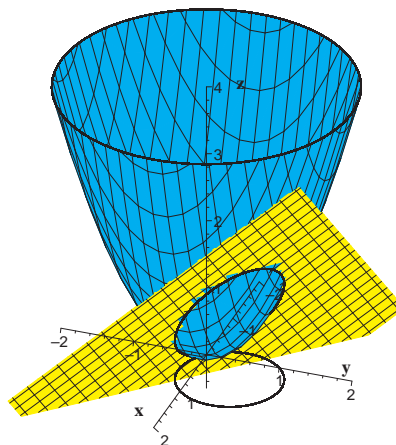
5. Izračunati zapreminu tela ograničenog površima $z_1 = z_1(x, y) = x^2 + y^2$ i $z_2 = z_2(x, y) = x + y$.

Rešenje. Projekcija preseka površi $z_1 = z_1(x, y)$ i $z_2 = z_2(x, y)$ predstavlja kružnicu:

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}.$$

Unutar kruga $D_{xy} : \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \leq \frac{1}{2}$ ravan $z_1 = z_1(x, y)$ je iznad parabolida $z_2 = z_2(x, y)$. Samim tim tražena zapremina određena je sa:

$$\begin{aligned} V &= \iint_{D_{xy}} (z_1(x, y) - z_2(x, y)) \, dx dy \\ &= \iint_{D_{xy}} (x + y - x^2 - y^2) \, dx dy \\ &= \iint_{D_{xy}} \left(\frac{1}{2} - \left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2\right) \, dx dy. \end{aligned}$$



Ako uvedemo polarne koordinate:

$$\left\{ \begin{array}{l} x - \frac{1}{2} = \rho \cos \varphi \\ y - \frac{1}{2} = \rho \sin \varphi \end{array} \right\}$$

krug D_{xy} je slika pravougaonika $D' : 0 \leq \rho \leq 1/\sqrt{2} \wedge 0 \leq \varphi \leq 2\pi$. Tada je $|J| = \rho$. Samim tim, posle uvođenja smene promenljivih, tražena zapremina data je vrednošću:

$$V = \iint_{D'} \left(\frac{1}{2} - \rho^2\right) |J| d\rho d\varphi = \int_0^{2\pi} \left(\int_0^{1/\sqrt{2}} \left(\frac{1}{2} - \rho^2\right) \rho d\rho \right) d\varphi = 2\pi \left(\frac{\rho^2}{4} - \frac{\rho^4}{4} \right) \Bigg|_{\rho=0}^{1/\sqrt{2}} = \frac{\pi}{8}.$$

6. Izračunati zapreminu tela ograničenog površima:

$$y^2 = x, \quad y^2 = 4x, \quad z = 0, \quad x + z = 4.$$

Rešenje. Za $y \geq 0$ formirajmo u Oxy ravni skup:

$$D_{x,y} = \{(x, y) \in \mathbf{R}^2 \mid \sqrt{x} \leq y \leq 2\sqrt{x} \wedge 0 \leq x \leq 4\}$$

Za funkciju $z = z(x, y) = 4 - x$ tražena zapremina data je integralom:

$$V = 2 \iint_{D_{x,y}} z \, dx dy.$$

Prema tome, tražena zapremina iznosi:

$$V = 2 \iint_{D_{x,y}} (4 - x) \, dx dy = 2 \int_0^4 \left((4 - x) \cdot \int_{\sqrt{x}}^{2\sqrt{x}} dy \right) dx = 2 \int_0^4 \left((4 - x) \cdot (y) \Bigg|_{y=\sqrt{x}}^{2\sqrt{x}} \right) dx = \frac{256}{15}.$$

7. Izračunati dvostruki integral: $I = \iint_D \frac{dx \, dy}{(1 + x^2 + y^2)^{3/2}}$, gde je $D = [0, 1] \times [0, 1]$.

Rešenje. Ako uvedemo polarne koordinate:

$$\left\{ \begin{array}{l} x = x(\rho, \varphi) = \rho \cos \varphi, \\ y = y(\rho, \varphi) = \rho \sin \varphi, \end{array} \right\}$$

na osnovu $|J| = \rho$, dobijamo:

$$I = \iint_{D'} \frac{|J| d\rho d\varphi}{(1 + \rho^2)^{3/2}} = \iint_{D'} \frac{\rho d\rho d\varphi}{(1 + \rho^2)^{3/2}},$$

gde je D' oblast u \mathbf{R}^2 koja se skoro svuda jednoznačno preslikava na oblast D . Odredimo oblast D' . Primitimo da se granica oblasti $D = [0, 1] \times [0, 1]$ sastoji se od sledećih skupova:

$$\begin{aligned}\mathcal{A} &= \{(x, y) \mid 0 < x \leq 1 \wedge y = 0\}, \\ \mathcal{B} &= \{(x, y) \mid x = 1 \wedge 0 \leq y < 1\}, \\ \mathcal{C} &= \{(x, y) \mid 0 < x \leq 1 \wedge y = 1\}, \\ \mathcal{D} &= \{(x, y) \mid x = 0 \wedge 0 < y \leq 1\}, \\ \mathcal{O} &= \{(0, 0)\}.\end{aligned}$$

Samim tim, preslikavanju polarnim koordinatama skupovi \mathcal{A} , \mathcal{B} , \mathcal{C} , \mathcal{D} i \mathcal{O} dobijaju se kao slike redom sledećih skupova:

$$\begin{aligned}\mathcal{A}' &= \{(\varphi, \rho) \mid 0 < \rho \leq 1 \wedge \varphi = 0\}, \\ \mathcal{B}' &= \{(\varphi, \rho) \mid \rho = \frac{1}{\cos \varphi} \wedge 0 < \varphi \leq \frac{\pi}{4}\}, \\ \mathcal{C}' &= \{(\varphi, \rho) \mid \rho = \frac{1}{\sin \varphi} \wedge \frac{\pi}{4} < \varphi \leq \frac{\pi}{2}\}, \\ \mathcal{D}' &= \{(\varphi, \rho) \mid 0 < \rho \leq 1 \wedge \varphi = \frac{\pi}{2}\}, \\ \mathcal{O}' &= \{(\varphi, \rho) \mid \rho = 0 \wedge 0 \leq \varphi \leq \frac{\pi}{2}\}.\end{aligned}$$

Prethodni skupovi ograničavaju oblast:

$$D' = \left\{ (\varphi, \rho) \mid \left(0 \leq \rho \leq \frac{1}{\cos \varphi} \wedge 0 \leq \varphi \leq \frac{\pi}{4} \right) \cup \left(0 \leq \rho \leq \frac{1}{\sin \varphi} \wedge \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2} \right) \right\},$$

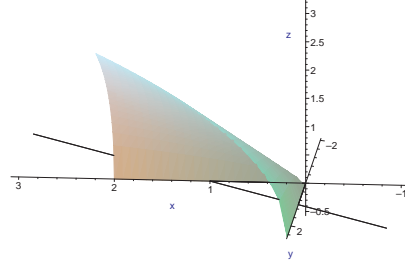
koja se polarnim koordinatama skoro svuda jednoznačno slika na oblast D . Posle određivanja oblasti D' računamo vrednost integrala:

$$\begin{aligned}I &= \iint_{D'} \frac{\rho d\rho d\varphi}{(1 + \rho^2)^{3/2}} = \int_0^{\pi/4} \left(\int_0^{1/\cos \varphi} \frac{\rho}{(1 + \rho^2)^{3/2}} d\rho \right) d\varphi + \int_{\pi/4}^{\pi/2} \left(\int_0^{1/\sin \varphi} \frac{\rho}{(1 + \rho^2)^{3/2}} d\rho \right) d\varphi \\ &= \frac{1}{2} \int_0^{\pi/4} \left(\int_0^{1/\cos \varphi} \frac{d(1 + \rho^2)}{(1 + \rho^2)^{3/2}} \right) d\varphi + \frac{1}{2} \int_{\pi/4}^{\pi/2} \left(\int_0^{1/\sin \varphi} \frac{d(1 + \rho^2)}{(1 + \rho^2)^{3/2}} \right) d\varphi = \frac{1}{2} \int_0^{\pi/4} \left(\frac{1}{\sqrt{1 + \rho^2}} \Big|_{\rho=0}^{1/\cos \varphi} \right) d\varphi \\ &+ \frac{1}{2} \int_{\pi/4}^{\pi/2} \left(\frac{1}{\sqrt{1 + \rho^2}} \Big|_{\rho=0}^{1/\sin \varphi} \right) d\varphi = \frac{1}{2} \int_0^{\pi/4} \left(1 - \frac{\cos \varphi}{\sqrt{1 + \cos^2 \varphi}} \right) d\varphi + \frac{1}{2} \int_{\pi/4}^{\pi/2} \left(1 - \frac{\sin \varphi}{\sqrt{1 + \sin^2 \varphi}} \right) d\varphi \\ &= \frac{1}{2} \int_0^{\pi/4} d\varphi - \frac{1}{2} \int_0^{\pi/4} \frac{d(\sin \varphi)}{\sqrt{2 - \sin^2 \varphi}} + \frac{1}{2} \int_{\pi/4}^{\pi/2} d\varphi - \frac{1}{2} \int_{\pi/4}^{\pi/2} \frac{d(\cos \varphi)}{\sqrt{2 - \cos^2 \varphi}} = \frac{\pi}{2} - \arcsin \left(\frac{\sin \varphi}{\sqrt{2}} \right) \Big|_{\varphi=0}^{\pi/4} \\ &+ \arcsin \left(\frac{\cos \varphi}{\sqrt{2}} \right) \Big|_{\varphi=\pi/4}^{\pi/2} = \frac{\pi}{2} - \arcsin \left(\frac{1}{2} \right) - \arcsin \left(\frac{1}{2} \right) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.\end{aligned}$$

8. Izračunati površinu dela površi $S : z^2 = 2xy$ određene u prvom oktantu u preseku sa ravnima: $x = 0$, $y = 0$ i $x + y = 1$.

Rešenje. U prvom oktantu posmatrajmo funkciju $z = z(x, y) = \sqrt{2xy}$. Neka je T trougao određen u Oxy ravni pravima $x = 0$, $y = 0$ i $x + y = 1$. Tada je površina površi data formulom:

$$P = \iint_S \sqrt{1 + z'_x(x, y)^2 + z'_y(x, y)^2} dx dy,$$



gde je S deo površi određen funkcijom $z = z(x, y)$ nad trouglom T . Samim tim iz parcijalnih izvoda:

$$z'_x = \frac{\sqrt{y}}{\sqrt{2x}} \quad \text{i} \quad z'_y = \frac{\sqrt{x}}{\sqrt{2y}}$$

nalazimo traženu površinu površi:

$$\begin{aligned} P &= \iint_S \frac{x+y}{\sqrt{2xy}} dx dy = \frac{1}{\sqrt{2}} \iint_T (x^{\frac{1}{2}}y^{-\frac{1}{2}} + x^{-\frac{1}{2}}y^{\frac{1}{2}}) dx dy \\ &= \frac{1}{\sqrt{2}} \int_0^1 \left(\int_0^{1-x} (x^{\frac{1}{2}}y^{-\frac{1}{2}} + x^{-\frac{1}{2}}y^{\frac{1}{2}}) dy \right) dx \\ &= \sqrt{2} \int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} dx + \frac{\sqrt{2}}{3} \int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{3}{2}} dx \\ &= \sqrt{2} \int_0^1 x^{\frac{3}{2}-1}(1-x)^{\frac{3}{2}-1} dx + \frac{\sqrt{2}}{3} \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{5}{2}-1} dx \\ &= \sqrt{2}B\left(\frac{3}{2}, \frac{3}{2}\right) + \frac{\sqrt{2}}{3}B\left(\frac{1}{2}, \frac{5}{2}\right) = \frac{\pi}{2\sqrt{2}}. \end{aligned}$$

9. Izračunati površinu i zapreminu tela ograničenog loptom:

$$x^2 + y^2 + z^2 = 3a^2$$

i paraboloidom:

$$x^2 + y^2 = 2az,$$

gde je $a > 0$.

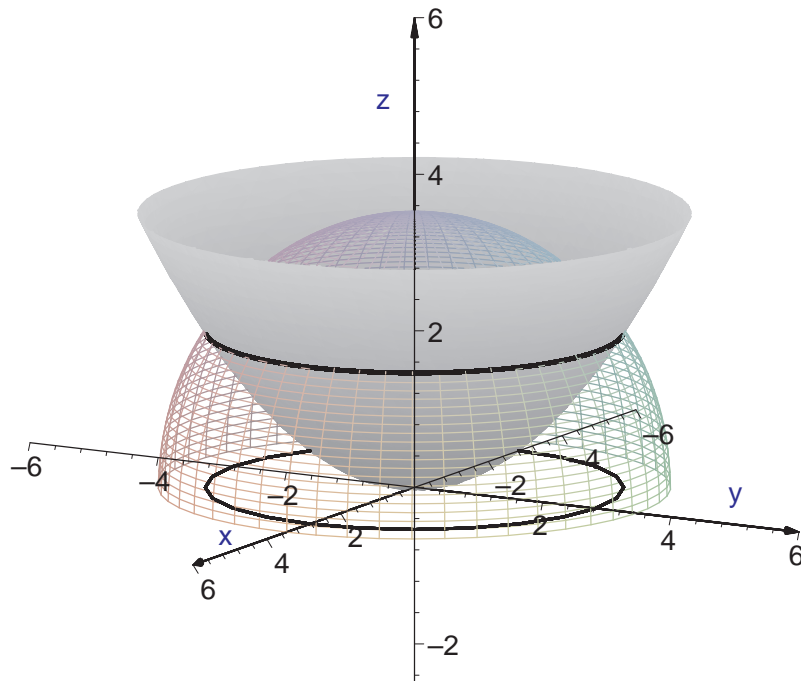
Rešenje. Posmatrano telo se dobija u preseku površi gornje polulopte:

$$(1) \quad z = f_1(x, y) = \sqrt{3a^2 - (x^2 + y^2)}$$

i paraboloida:

$$(2) \quad z = f_2(x, y) = \frac{1}{2a}(x^2 + y^2).$$

Eliminacijom $x^2 + y^2$ iz (1) i (2) dobijamo kvadratnu jednačinu $z^2 + 2az - 3a^2 = 0$. Budući da je $z > 0$ zaključujemo da je presek posmatrana dva tela kružnica: $x^2 + y^2 = 2a^2$ na visini $z = a$. Posmatrano telo se projektuje na krug $D_{xy} : x^2 + y^2 \leq 2a^2$.



(i) Odredimo površinu tela ograničenog delom lopte i delom paraboloida. Ako označimo površinu dela lopte sa P_1 tada je:

$$P_1 = \iint_{D_{xy}} \sqrt{1 + \left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2} dx dy = \sqrt{3}a \iint_{D_{xy}} \frac{1}{\sqrt{3a^2 - x^2 - y^2}} dx dy.$$

Ako označimo površinu dela paraboloida sa P_2 tada je:

$$P_2 = \iint_{D_{xy}} \sqrt{1 + \left(\frac{\partial f_2}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2} dx dy = \frac{1}{a} \iint_{D_{xy}} \sqrt{a^2 + x^2 + y^2} dx dy.$$

Samim tim tražena površina celog tela je:

$$P = \sqrt{3}a \iint_{D_{xy}} \frac{1}{\sqrt{3a^2 - x^2 - y^2}} dx dy + \frac{1}{a} \iint_{D_{xy}} \sqrt{a^2 + x^2 + y^2} dx dy.$$

Uvođenjem polarnih koordinata: $x = \rho \cos \varphi$, $y = \rho \sin \varphi$ ($|J| = \rho$) oblast $D_{\rho\varphi}$: $0 \leq \rho \leq \sqrt{2}a$ \wedge $0 \leq \varphi \leq 2\pi$ se skoro svuda jednoznačno preslikava na oblast D_{xy} . Na osnovu toga:

$$\begin{aligned} P &= \sqrt{3}a \iint_{D_{\rho\varphi}} \frac{1}{\sqrt{3a^2 - \rho^2}} \underbrace{\rho}_{|J|} d\rho d\varphi + \frac{1}{a} \iint_{D_{\rho\varphi}} \sqrt{a^2 + \rho^2} \underbrace{\rho}_{|J|} d\rho d\varphi \\ &= \sqrt{3}a \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}a} \frac{\rho}{\sqrt{3a^2 - \rho^2}} d\rho + \frac{1}{a} \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}a} \sqrt{a^2 + \rho^2} \rho d\rho \\ &= 2\sqrt{3}\pi a \left(-\sqrt{3a^2 - \rho^2} \right) \Big|_0^{\sqrt{2}a} + \frac{2\pi}{a} \left((a^2 + \rho^2)^{3/2} \right) \Big|_0^{\sqrt{2}a} \\ &= 2\sqrt{3}\pi a (-a + \sqrt{3}a) + \frac{2\pi}{a} (3\sqrt{3}a^3 - a^2) = \frac{16}{3}a^2\pi. \end{aligned}$$

(ii) Odredimo zapreminu $\mu = \mu(V)$ tela V ograničenog delom lopte i delom paraboloida:

$$\begin{aligned}\mu &= \iiint_V 1 \, dx dy dz = \iint_{D_{xy}} \left(f_1(x, y) - f_2(x, y) \right) dx dy \\ &= \iint_{D_{xy}} \left(\sqrt{3a^2 - x^2 - y^2} - \frac{1}{2a}(x^2 + y^2) \right) dx dy.\end{aligned}$$

Uvođenjem polarnih koordinata: $x = \rho \cos \varphi$, $y = \rho \sin \varphi$ ($|J| = \rho$) oblast $D_{\rho\varphi}$: $0 \leq \rho \leq \sqrt{2}a$ \wedge $0 \leq \varphi \leq 2\pi$ se skoro svuda jednoznačno preslikava na oblast D_{xy} . Na osnovu toga:

$$\begin{aligned}\mu &= \iint_{D_{\rho\varphi}} \left(\sqrt{3a^2 - \rho^2} - \frac{1}{2a}\rho^2 \right) \underbrace{\rho}_{|J|} d\rho d\varphi = \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}a} \left(\sqrt{3a^2 - \rho^2} \rho - \frac{1}{2a}\rho^3 \right) d\rho \\ &= 2\pi \left(-\frac{1}{3}(3a^2 - \rho^2)^{3/2} - \frac{1}{8a}\rho^4 \right) \Big|_0^{\sqrt{2}a} = 2\pi \left(\left(-\frac{a^3}{3} + \sqrt{3}a^3 \right) - \frac{a^3}{2} \right) = \frac{a^3\pi}{3}(6\sqrt{3} - 5).\end{aligned}$$

10. Izračunati površinski integral:

$$\iint_{S^+} x^2 \, dy dz + y^2 \, dz dx + z^2 \, dx dy,$$

gde je S^+ spoljašnja strana kupe određene omotačem $z^2 = x^2 + y^2$, $0 \leq z \leq h$ i osnovom $x^2 + y^2 \leq h^2$, $z = h$ za fiksirano $h > 0$.

Rešenje. I Način. Označimo sa V unutrašnjost kupe koju obuhvata površ S . Funkcije:

$$P = x^2, \quad Q = y^2, \quad R = z^2 : V \longrightarrow \mathbf{R}$$

ispunjavaju uslove za primenu teoreme OSTROGRADSKOG. Samim tim:

$$I = \iint_{S^+} x^2 \, dy dz + y^2 \, dz dx + z^2 \, dx dy = \iiint_V (2x+2y+2z) \, dx dy dz.$$

Uvedimo cilindrične koordinate: $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, $z = z$ za vrednosti $(\rho, \varphi, z) \in V_1 = \{(\rho, \varphi, z) \mid 0 \leq \varphi \leq 2\pi \wedge \rho \leq z \leq h : \rho \in [0, h]\}$. Tada je $|J| = \rho$. Uvođenjem cilindričnih koordinata dobijamo:

$$\begin{aligned}I &= \iiint_{V_1} (2\rho \cos \varphi + 2\rho \sin \varphi + 2z)\rho \, dz d\rho d\varphi \\ &= 2 \int_0^{2\pi} \int_0^h \int_\rho^h (\rho^2(\cos \varphi + \sin \varphi) + \rho z) \, dz d\rho d\varphi \\ &= 2 \int_0^{2\pi} \int_0^h (\rho^2(\cos \varphi + \sin \varphi)(h - \rho) + \rho \frac{h^2 - \rho^2}{2}) \, d\rho d\varphi \\ &= 2 \int_0^{2\pi} \int_0^h ((\rho^2 h - \rho^3)(\cos \varphi + \sin \varphi) + \frac{h^2}{2}\rho - \frac{\rho^3}{2}) \, d\rho d\varphi \\ &= 2 \int_0^{2\pi} \left(\left(\frac{h^4}{3} - \frac{h^4}{4} \right) (\cos \varphi + \sin \varphi) + \frac{h^4}{4} - \frac{h^4}{8} \right) d\varphi = 2 \int_0^{2\pi} \frac{h^4}{8} d\varphi = \frac{h^4\pi}{2}.\end{aligned}$$

II Način. Integral:

$$I = \iint_{S^+} x^2 dydz + y^2 dzdx + z^2 dxdy,$$

rastavimo na zbir dva površinska integrala:

$$I_1 = \iint_{M^-} x^2 dydz + y^2 dzdx + z^2 dxdy$$

i

$$I_2 = \iint_{B^+} x^2 dydz + y^2 dzdx + z^2 dxdy,$$

gde je M omotač kupe i B baza kupe. Za računanje prvog integrala uvedimo smenu promenljivih:

$$\begin{aligned} x &= x(\varphi, \rho) = \rho \cos \varphi, \\ y &= y(\varphi, \rho) = \rho \sin \varphi, \\ z &= z(\varphi, \rho) = \rho; \end{aligned}$$

za $(\varphi, \rho) \in D_{\varphi\rho} = [0, 2\pi] \times [0, h]$. Tada nalazimo redom parcijalne izvode:

$$\begin{aligned} x'_\varphi &= -\rho \sin \varphi, & x'_\rho &= \cos \varphi, \\ y'_\varphi &= \rho \cos \varphi, & y'_\rho &= \sin \varphi, \\ z'_\varphi &= 0, & z'_\rho &= 1. \end{aligned}$$

Formirajmo odgovarajuće determinante:

$$A = \begin{vmatrix} y'_\varphi & z'_\varphi \\ y'_\rho & z'_\rho \end{vmatrix} = \rho \cos \varphi, \quad B = \begin{vmatrix} z'_\varphi & x'_\varphi \\ z'_\rho & x'_\rho \end{vmatrix} = \rho \sin \varphi, \quad C = \begin{vmatrix} x'_\varphi & y'_\varphi \\ x'_\rho & y'_\rho \end{vmatrix} = -\rho.$$

Samim tim, na osnovu položaja vektora normale površine $\vec{n} = (A, B, C)$, zaključujemo:

$$\begin{aligned} I_1 &= \iint_{M^-} x^2 dydz + y^2 dzdx + z^2 dxdy, \\ &= \iint_{D_{\varphi\rho}} \left((\rho \cos \varphi)^2 \cdot \underbrace{(\rho \cos \varphi)}_A + (\rho \sin \varphi)^2 \cdot \underbrace{(\rho \sin \varphi)}_B + (\rho)^2 \cdot \underbrace{(-\rho)}_C \right) d\varphi d\rho \\ &= \int_0^{2\pi} \left(\int_0^h \rho^3 (\cos^3 \varphi + \sin^3 \varphi - 1) d\rho \right) d\varphi \\ &= \frac{h^4}{4} \int_0^{2\pi} (\cos^3 \varphi + \sin^3 \varphi - 1) d\varphi = -\frac{h^4 \pi}{2}. \end{aligned}$$

Sa druge strane za bazu B važi $z = h$ i odatle očigledno:

$$I_2 = \iint_{B^+} z^2 dxdy = \iint_{B_{xy}} h^2 dxdy = h^2 \iint_{B_{xy}} 1 dxdy = h^4 \pi,$$

gde je $B_{xy} : x^2 + y^2 \leq h^2$. Sveukupno:

$$I = I_1 + I_2 = \frac{h^4 \pi}{2}.$$

11. Izračunati površinski integral:

$$I = \iint_{S^+} y^2 dydz + (y^2 + x^2) dzdx + (y^2 + x^2 + z^2) dxdy,$$

gde je S^+ spoljašnja strana polusfere: $x^2 + y^2 + z^2 = 2Rx$, $z > 0$ (za fiksirano $R > 0$).

Rešenje. I Način. Primenimo formulu OSTROGRADSKOG. Zatvorimo datu površ S delom ravni:

$$(S_0^-) \quad (x^2 - R)^2 + y^2 \leq R^2 \wedge z = 0,$$

uzimajući za jedinični vektor normale $\vec{n}_0 = (0, 0, -1)$. Budući da je: $\frac{\partial}{\partial x}y^2 + \frac{\partial}{\partial y}(x^2 + y^2) + \frac{\partial}{\partial z}(y^2 + x^2 + z^2) = 2y + 2z$, saglasno formuli OSTROGRADSKOG, umesto polaznog površinskog integrala I formirajmo:

$$I_1 = \iint_{S^+ \cup S_0^-} y^2 dydz + (y^2 + x^2) dzdx + (y^2 + x^2 + z^2) dxdy = \iiint_V (2y + 2z) dxdydz,$$

gde je $V = \{(x, y, z) \in \mathbf{R}^3 \mid 0 \leq z \leq \sqrt{2Rx - x^2 - y^2} \wedge (x, y) \in D_{xy}\}$ i gde je $D_{xy} = \{(x, y) \in \mathbf{R}^2 \mid (x - R)^2 + y^2 \leq R^2\}$. Dalje, označimo:

$$I_2 = \iint_{S_0^-} y^2 dydz + (y^2 + x^2) dzdx + (y^2 + x^2 + z^2) dxdy,$$

pri čemu je S_0^- uzeto sa negativne strane dela ravni $z = 0$. Tada važi:

$$I = I_1 - I_2 = \iiint_V (2y + 2z) dxdydz - \iint_{S_0^-} y^2 dydz + (y^2 + x^2) dzdx + (y^2 + x^2 + z^2) dxdy$$

Izračunajmo I_1 . Uvedimo sferne koordinate:

$$x = R + \rho \cos \theta \cos \varphi, \quad y = \rho \cos \theta \sin \varphi, \quad z = \rho \sin \theta$$

za $(\varphi, \theta, \rho) \in [0, 2\pi] \times [0, \frac{\pi}{2}] \times [0, R]$. Tada je $|J| = \rho^2 \cos \theta$. Uvođenjem sfernih koordinata dobijamo:

$$\begin{aligned} I_1 &= \iiint_V (2y + 2z) dxdydz = \int_0^{2\pi} \int_0^{\pi/2} \int_0^R (2\rho \cos \theta \sin \varphi + 2\rho \sin \theta) \cdot \rho^2 \cos \theta d\rho d\theta d\varphi \\ &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \rho^3 (2 \sin \varphi \cos^2 \theta + 2 \sin \theta \cos \theta) d\rho d\theta d\varphi \\ &= \frac{R^4}{4} \int_0^{2\pi} \int_0^{\pi/2} (2 \sin \varphi \cdot \cos^2 \theta + 2 \sin \theta \cos \theta) d\theta d\varphi \\ &= \frac{R^4}{4} \int_0^{2\pi} \int_0^{\pi/2} \left(2 \sin \varphi \cdot \frac{1 + \cos 2\theta}{2} + 2 \sin \theta \cos \theta \right) d\theta d\varphi \\ &= \frac{R^4}{4} \int_0^{2\pi} \left(2 \sin \varphi \cdot \frac{\theta + \sin \theta \cos \theta}{2} + 2 \frac{\sin^2 \theta}{2} \right) \Big|_{\theta=0}^{\pi/2} d\varphi \\ &= \frac{R^4}{4} \int_0^{2\pi} \left(\frac{\pi}{2} \sin \varphi + 1 \right) d\varphi = \frac{\pi R^4}{2}. \end{aligned}$$

Izračunajmo I_2 (površinski integral po S_0^-). Uočimo da važi:

$$I_2 = \iint_{S_0^-} y^2 dydz + (y^2 + x^2) dzdx + (y^2 + x^2 + z^2) dxdy = - \iint_{D_{xy}} (x^2 + y^2) dxdy,$$

jer je S_0^- uzeto sa negativne strane dela ravni $z = 0$. Uvedimo polarne koordinate:

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi,$$

za $(\rho, \varphi) \in D_{\rho\varphi} = \{(\rho, \varphi) \in \mathbf{R}^2 \mid 0 \leq \rho \leq 2R \cos \varphi \wedge -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}\}$. Tada je $|J| = \rho$.

Uvođenjem polarnih koordinata dobijamo:

$$\begin{aligned} I_2 &= - \iint_{D_{\rho\varphi}} (\rho^2) \rho d\rho d\varphi = - \iint_{D_{\rho\varphi}} \rho^3 d\rho d\varphi = - \int_{-\pi/2}^{\pi/2} \left(\int_0^{2R \cos \varphi} \rho^3 d\rho \right) d\varphi \\ &= -4R^4 \int_{-\pi/2}^{\pi/2} \cos^4 \varphi d\varphi = -8R^4 \int_0^{\pi/2} \cos^4 \varphi \sin^0 \varphi d\varphi \\ &= -4R^4 \cdot B\left(\frac{4+1}{2}, \frac{0+1}{2}\right) = -4R^4 \cdot \frac{3\pi}{8} = -\frac{3\pi R^4}{2}. \end{aligned}$$

$$\text{Sveukupno: } I = I_1 - I_2 = \frac{\pi R^4}{2} - \left(-\frac{3\pi R^4}{2}\right) = 2\pi R^4.$$

II Način. Zadatak možemo rešiti i smenom promenljivih:

$$\begin{aligned} x &= x(\varphi, \theta) = R + R \cos \theta \cos \varphi, \\ y &= y(\varphi, \theta) = R \cos \theta \sin \varphi, \\ z &= z(\varphi, \theta) = R \sin \theta; \end{aligned}$$

za $(\varphi, \theta) \in D_{\varphi, \theta} = [0, 2\pi] \times [0, \pi/2]$. Tada nalazimo redom parcijalne izvode:

$$\begin{aligned} x'_\varphi &= -R \sin \varphi \cos \theta, & x'_\theta &= -R \cos \varphi \sin \theta, \\ y'_\varphi &= R \cos \varphi \cos \theta, & y'_\theta &= -R \sin \varphi \sin \theta, \\ z'_\varphi &= 0, & z'_\theta &= R \cos \theta. \end{aligned}$$

Formirajmo odgovarajuće determinante:

$$A = \begin{vmatrix} y'_\varphi & z'_\varphi \\ y'_\theta & z'_\theta \end{vmatrix} = R^2 \cos \varphi \cos^2 \theta, \quad B = \begin{vmatrix} z'_\varphi & x'_\varphi \\ z'_\theta & x'_\theta \end{vmatrix} = R^2 \sin \varphi \cos^2 \theta, \quad C = \begin{vmatrix} x'_\varphi & y'_\varphi \\ x'_\theta & y'_\theta \end{vmatrix} = R^2 \cos \theta \sin \theta.$$

Samim tim, na osnovu položaja vektora normale površine $\vec{n} = (A, B, C)$, zaključujemo:

$$\begin{aligned} I &= \iint_S y^2 dydz + (y^2 + x^2) dzdx + (y^2 + x^2 + z^2) dxdy \\ &= \iint_{D_{\varphi\theta}} \left(\left[(R \sin \varphi \cos \theta)^2 \cdot \underbrace{(R^2 \cos \varphi \cos^2 \theta)}_A \right] \right. \\ &\quad + \left[\left((R \sin \varphi \cos \theta)^2 + (R + R \cos \varphi \cos \theta)^2 \right) \cdot \underbrace{(R^2 \sin \varphi \cos^2 \theta)}_B \right] \\ &\quad \left. + \left[\left((R \sin \varphi \cos \theta)^2 + (R + R \cos \varphi \cos \theta)^2 + (R \sin \theta)^2 \right) \cdot \underbrace{(R^2 \cos \theta \sin \theta)}_C \right] \right) d\varphi d\theta \end{aligned}$$

Odatle, računamo vrednost traženog integrala:

$$\begin{aligned}
 I &= R^4 \int_0^{\pi/2} \left(\int_0^{2\pi} \left(\left[\sin^2 \varphi \cos \varphi \cos^4 \theta \right] + \left[\sin^3 \varphi \cos^4 \theta + \sin \varphi \cos^2 \theta + 2 \sin \varphi \cos \varphi \cos^3 \theta \right. \right. \right. \\
 &+ \left. \left. \left. \left[\sin \varphi \cos^2 \varphi \cos^4 \theta \right] + \left[\sin^2 \varphi \sin \theta \cos^3 \theta + \sin \theta \cos \theta + 2 \cos \varphi \sin \theta \cos^2 \theta + \cos^2 \varphi \sin \theta \cos^3 \theta \right. \right. \right. \\
 &+ \left. \left. \left. \left[\sin^3 \theta \cos \theta \right] \right) d\varphi \right) d\theta = 2R^4 \pi \int_0^{\pi/2} \left(\sin \theta \cos^3 \theta + \sin \theta \cos \theta + \sin^3 \theta \cos \theta \right) d\theta \\
 &= \left(B(1,2) + B(1,1) + B(2,1) \right) \pi R^4 = 2\pi R^4.
 \end{aligned}$$

12. Neka je data je kriva koordinatama:

$$x = a \cos^2 t, \quad y = a \sin t \cos t, \quad z = a \sin t$$

za $a > 0$ i $t \in [0, 2\pi]$. Pokazati da ova kriva leži u preseku jedne sfere i jednog cilindra i odrediti jednačine tih površi. Odrediti površinu koju cilindar iseca na lopti iznad ravni Oxy i odrediti zapreminu tela ograničenog loptom, cilindrom i ravni Oxy u gornjem poluprostoru.

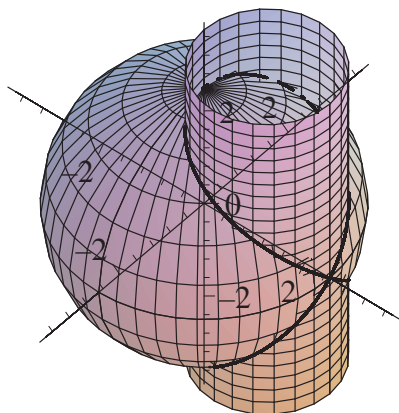
Rešenje. (i) Važi:

$$\begin{aligned}
 x^2 + y^2 + z^2 &= a^2 \cos^4 t + a^2 \sin^2 t \cos^2 t + a^2 \sin^2 t \\
 &= a^2 \cos^2 t \cdot (\cos^2 t + \sin^2 t) + a^2 \sin^2 t = a^2
 \end{aligned}$$

i

$$\begin{aligned}
 x^2 + y^2 &= a^2 \cos^4 t + a^2 \sin^2 t \cos^2 t \\
 &= a^2 \cos^2 t \cdot (\cos^2 t + \sin^2 t) = a^2 \cos^2 t = ax.
 \end{aligned}$$

Samim tim posmatrana kriva nalazi se u preseku sfere: $x^2 + y^2 + z^2 = a^2$ i cilindra: $(x - a/2)^2 + y^2 = (a/2)^2$.



(ii) Površ čiju površinu računamo određena je jednačinom gornje polusfere: $z = z(x, y) = \sqrt{a^2 - x^2 - y^2}$ za $(x, y) \in D = \{(x, y) \mid (x - a/2)^2 + y^2 \leq (a/2)^2\}$. Tražena površina se računa po obrascu:

$$P = \iint_D \sqrt{1 + z'_x{}^2(x, y) + z'_y{}^2(x, y)} dx dy,$$

gde je:

$$z'_x = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}, \quad z'_y = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}.$$

Samim tim tražena površina data je integralom:

$$P = \iint_D \frac{a \, dx \, dy}{\sqrt{a^2 - (x^2 + y^2)}}.$$

Uvedimo polarne koordinate:

$$(*) \quad x = a\rho \cos \varphi \quad \wedge \quad y = a\rho \sin \varphi$$

za $(\rho, \varphi) \in D_1 = \{(\rho, \varphi) \mid 0 \leq \rho \leq \cos \varphi \quad \wedge \quad -\pi/2 \leq \varphi \leq \pi/2\}$. Tada je $|J| = a^2 \rho$. Odatle prethodni integral, posle uvođenja novih promenljivih, ima vrednost:

$$\begin{aligned} P &= \iint_{D_1} \frac{|J| \, d\rho \, d\varphi}{\sqrt{1 - \rho^2}} = 2a^2 \int_0^{\pi/2} \int_0^{\cos \varphi} \frac{\rho \, d\rho}{\sqrt{1 - \rho^2}} \, d\varphi = 2a^2 \int_0^{\pi/2} \left(-\sqrt{1 - \rho^2} \right) \Big|_{\rho=0}^{\cos \varphi} \, d\varphi \\ &= 2a^2 \int_0^{\pi/2} (1 - \sin \varphi) \, d\varphi = 2a^2 \left(\frac{\pi}{2} - 1 \right) = (\pi - 2) \cdot a^2. \end{aligned}$$

(iii) Pri prethodno uvedenim oznakama tražena zapremina se računa pomoću formule:

$$V = \iiint_D \sqrt{a^2 - (x^2 + y^2)} \, dx \, dy.$$

Tada, prelaskom na polarne koordinate (*), nalazimo:

$$\begin{aligned} V &= a \iint_{D_1} \sqrt{1 - \rho^2} |J| \, d\rho \, d\varphi = 2a^3 \int_0^{\pi/2} \left(\int_0^{\cos \varphi} \sqrt{1 - \rho^2} \rho \, d\rho \right) \, d\varphi \\ &= -a^3 \int_0^{\pi/2} \left(\int_0^{\cos \varphi} \sqrt{1 - \rho^2} \, d(1 - \rho^2) \right) \, d\varphi = -\frac{2a^3}{3} \int_0^{\pi/2} \left((1 - \rho^2)^{3/2} \right) \Big|_{\rho=0}^{\cos \varphi} \, d\varphi \\ &= -\frac{2a^3}{3} \int_0^{\pi/2} (\sin^3 \varphi - 1) \, d\varphi = \frac{2a^3}{3} \int_0^{\pi/2} d\varphi - \frac{2a^3}{3} \cdot \int_0^{\pi/2} \cos^0 \varphi \sin^3 \varphi \, d\varphi \\ &= \frac{a^3 \pi}{3} - \frac{2a^3}{3} \cdot \frac{1}{2} B\left(\frac{0+1}{2}, \frac{3+1}{2}\right) = \left(\frac{\pi}{3} - \frac{4}{9}\right) \cdot a^3. \end{aligned}$$

13. Izračunati:

$$I = \iint_{S^+} \left(\frac{dy \, dz}{x} + \frac{dz \, dx}{y} + \frac{dx \, dy}{z} \right),$$

gde je S^+ spoljašnja strana jedinične sfere.

Rešenje. I Način. Zadatak možemo rešiti smenom promenljivih:

$$\begin{aligned} x &= x(\varphi, \theta) = \cos \theta \cos \varphi, \\ y &= y(\varphi, \theta) = \cos \theta \sin \varphi, \\ z &= z(\varphi, \theta) = \sin \theta; \end{aligned}$$

za $(\varphi, \theta) \in D_{\varphi\theta} = [0, 2\pi] \times [-\pi/2, \pi/2]$. Tada nalazimo redom parcijalne izvode:

$$\begin{aligned}x'_\varphi &= -\sin \varphi \cos \theta, & x'_\theta &= -\cos \varphi \sin \theta, \\y'_\varphi &= \cos \varphi \cos \theta, & y'_\theta &= -\sin \varphi \sin \theta, \\z'_\varphi &= 0; & z'_\theta &= \cos \theta.\end{aligned}$$

Formirajmo odgovarajuće determinante:

$$A = \begin{vmatrix} y'_\varphi & z'_\varphi \\ y'_\theta & z'_\theta \end{vmatrix} = \cos \varphi \cos^2 \theta, \quad B = \begin{vmatrix} z'_\varphi & x'_\varphi \\ z'_\theta & x'_\theta \end{vmatrix} = \sin \varphi \cos^2 \theta, \quad C = \begin{vmatrix} x'_\varphi & y'_\varphi \\ x'_\theta & y'_\theta \end{vmatrix} = \cos \theta \sin \theta.$$

Samim tim, na osnovu položaja vektora normale površine $\vec{n} = (A, B, C)$, zaključujemo:

$$\begin{aligned}I &= \iint_{S^+} \frac{1}{x} dydz + \frac{1}{y} dzdx + \frac{1}{z} dxdy \\ &= \iint_{D_{\varphi\theta}} \left(\left[\frac{1}{\cos \theta \cos \varphi} \cdot \underbrace{\cos \varphi \cos^2 \theta}_A \right] + \left[\frac{1}{\cos \theta \sin \varphi} \cdot \underbrace{\sin \varphi \cos^2 \theta}_B \right] + \left[\frac{1}{\sin \theta} \cdot \underbrace{\cos \theta \sin \theta}_C \right] \right) d\varphi d\theta \\ &= \iint_{D_{\varphi\theta}} \left(\cos \theta + \cos \theta + \cos \theta \right) d\varphi d\theta = 3 \cdot \int_0^{2\pi} d\varphi \cdot \int_{-\pi/2}^{\pi/2} \cos \theta d\theta = 12\pi.\end{aligned}$$

II Način. Izvršimo dekompoziciju:

$$I = I_1 + I_2 + I_3,$$

gde je:

$$I_1 = \iint_{S^+} \frac{dydz}{x}, \quad I_2 = \iint_{S^+} \frac{dzdx}{y}, \quad I_3 = \iint_{S^+} \frac{dxdy}{z}.$$

Evidentno je da važi: $I_1 = I_2 = I_3$ i odatle $I = 3I_3$. Izračunajmo integral I_3 . Formirajmo funkcije:

$$z_1 = z_1(x, y) = \sqrt{1-x^2-y^2}, \quad z_2 = z_2(x, y) = -\sqrt{1-x^2-y^2} : D_{xy} \longrightarrow \mathbf{R}$$

sa domenom $D_{xy} = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$. Označimo sa S_1 gornju polusferu i sa S_2 donju polusferu. Tada važi:

$$I_3 = \iint_{S^+} \frac{dxdy}{z} = \iint_{S_1^+} \frac{dxdy}{z} + \iint_{S_2^-} \frac{dxdy}{z} = 2 \iint_{S_1^+} \frac{dxdy}{z},$$

jer važi:

$$\iint_{S_2^-} \frac{dxdy}{z} = - \iint_{D_{xy}} \frac{dxdy}{z_2(x, y)} = \iint_{D_{xy}} \frac{dxdy}{z_1(x, y)} = \iint_{S_1^+} \frac{dxdy}{z}.$$

Samim tim:

$$I = 3I_3 = 6 \iint_{D_{xy}} \frac{dxdy}{\sqrt{1-x^2-y^2}}.$$

Uvedimo polarne koordinate $x = \rho \cos \varphi \wedge y = \rho \sin \varphi$ za vrednosti $(\rho, \varphi) \in D_{\varphi\rho} = \{(\rho, \varphi) \mid 0 \leq \rho \leq 1 \wedge 0 \leq \varphi \leq 2\pi\}$. Tada je $|J| = \rho$. Samim tim prethodni integral, posle uvođenja novih promenljivih, ima vrednost:

$$I = 6 \int \int_{D_{\varphi\rho}} \frac{|J| d\varphi d\rho}{\sqrt{1-\rho^2}} = 6 \cdot \int_0^{2\pi} d\varphi \cdot \int_0^1 \frac{\rho}{\sqrt{1-\rho^2}} = 6 \cdot 2\pi \cdot \left(-\sqrt{1-\rho^2} \right) \Big|_0^1 = 12\pi.$$

Napomena. Funkcije:

$$P(x, y, z) = \frac{1}{x}, \quad Q(x, y, z) = \frac{1}{y}, \quad R(x, y, z) = \frac{1}{z} : \mathbf{R}^3 \setminus \{Ox \cup Oy \cup Oz\} \longrightarrow \mathbf{R}$$

u unutrašnjosti jedinične sfere ne ispunjavaju uslov za primenu teoreme OSTROGRADSKOG.

14. Neka su date redom funkcije $f : \mathbf{R}^3 \longrightarrow \mathbf{R}$ i $\vec{f} : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$. Ako je \vec{r} radijus vektor i $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ – LAPLACEov operator dokazati da važe sledeće jednakosti:

$$\begin{aligned} \Delta(x \cdot f) &= 2 \frac{\partial f}{\partial x} + x \cdot \Delta(f), \\ \text{grad}(f) &= \frac{1}{2} \left(\Delta(\vec{r} \cdot f) - \vec{r} \cdot \Delta(f) \right), \\ \text{div}(\vec{f}) &= \frac{1}{2} \left(\Delta(\vec{r} \circ \vec{f}) - \vec{r} \circ \Delta(\vec{f}) \right), \\ \text{rot}(\vec{f}) &= \frac{1}{2} \left(\Delta(\vec{r} \times \vec{f}) - \vec{r} \times \Delta(\vec{f}) \right). \end{aligned}$$

Rešenje. 1^o. Dokazaćemo prvu jednakost i na osnovu nje dokazujemo da važe i ostale tri jednakosti. Napomenimo da se sve navedene jednakosti mogu direktno dokazati izračunavanjem obe strane navedenih jednakosti. Za skalarnu funkciju f važi jednakost:

$$(*) \quad \Delta(x \cdot f) = 2 \frac{\partial f}{\partial x} + x \cdot \Delta(f).$$

Zaista:

$$\Delta(x \cdot f) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (x \cdot f) = \frac{\partial}{\partial x} \left(f + x \frac{\partial f}{\partial x} \right) + x \frac{\partial^2 f}{\partial y^2} + x \frac{\partial^2 f}{\partial z^2} = 2 \frac{\partial f}{\partial x} + x \cdot \Delta(f).$$

2^o. Važi:

$$\begin{aligned} \Delta(\vec{r} \cdot f(\vec{r})) &= (\Delta(xf), \Delta(yf), \Delta(zf)) \\ &\stackrel{(*)}{=} \left(2 \frac{\partial f}{\partial x} + x \cdot \Delta(f), 2 \frac{\partial f}{\partial y} + y \cdot \Delta(f), 2 \frac{\partial f}{\partial z} + z \cdot \Delta(f) \right) \\ &= 2 \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) + (x, y, z) \cdot \Delta(f(\vec{r})) = 2 \text{grad}(f(\vec{r})) + \vec{r} \cdot \Delta(f(\vec{r})). \end{aligned}$$

Odatle sleduje jednakost:

$$\text{grad}(f) = \frac{1}{2} \left(\Delta(\vec{r} \cdot f) - \vec{r} \cdot \Delta(f) \right).$$

3⁰. Važi:

$$\begin{aligned}
 \Delta(\vec{r} \circ \vec{f}(\vec{r})) &= \Delta(xf_1 + yf_2 + zf_3) = \Delta(xf_1) + \Delta(yf_2) + \Delta(zf_3) \\
 &\stackrel{(*)}{=} \left(2\frac{\partial f_1}{\partial x} + x\Delta(f_1)\right) + \left(2\frac{\partial f_2}{\partial y} + y\Delta(f_2)\right) + \left(2\frac{\partial f_3}{\partial z} + z\Delta(f_3)\right) \\
 &= 2\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) + (x, y, z) \circ (\Delta(f_1), \Delta(f_2), \Delta(f_3)) \\
 &= 2\operatorname{div}(\vec{f}(\vec{r})) + \vec{r} \circ \Delta(\vec{f}(\vec{r})).
 \end{aligned}$$

Odatle sleduje jednakost:

$$\operatorname{div}(\vec{f}) = \frac{1}{2} \left(\Delta(\vec{r} \circ \vec{f}) - \vec{r} \circ \Delta(\vec{f}) \right).$$

4⁰. Važi:

$$\begin{aligned}
 \Delta(\vec{r} \times \vec{f}(\vec{r})) &= \Delta \left(\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ f_1 & f_2 & f_3 \end{vmatrix} \right) = \Delta(yf_3 - zf_2, zf_1 - xf_3, xf_2 - yf_1) \\
 &= (\Delta(yf_3) - \Delta(zf_2), \Delta(zf_1) - \Delta(xf_3), \Delta(xf_2) - \Delta(yf_1)) \\
 &\stackrel{(*)}{=} \left(2\frac{\partial f_3}{\partial y} + y\Delta(f_3) - 2\frac{\partial f_2}{\partial z} - z\Delta(f_2), 2\frac{\partial f_1}{\partial z} + z\Delta(f_1) \right. \\
 &\quad \left. - 2\frac{\partial f_3}{\partial x} - x\Delta(f_3), 2\frac{\partial f_2}{\partial x} + x\Delta(f_2) - 2\frac{\partial f_1}{\partial y} - y\Delta(f_1) \right) \\
 &= 2 \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \\
 &\quad + (y\Delta(f_3) - z\Delta(f_2), z\Delta(f_1) - x\Delta(f_3), x\Delta(f_2) - y\Delta(f_1),) \\
 &= 2\operatorname{rot}(\vec{f}(\vec{r})) + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ \Delta(f_1) & \Delta(f_2) & \Delta(f_3) \end{vmatrix} = 2\operatorname{rot}(\vec{f}(\vec{r})) + \vec{r} \times \Delta(\vec{f}(\vec{r})).
 \end{aligned}$$

Odatle sleduje jednakost:

$$\operatorname{rot}(\vec{f}) = \frac{1}{2} \left(\Delta(\vec{r} \times \vec{f}) - \vec{r} \times \Delta(\vec{f}) \right).$$

15. Neka funkcije $g, h : \mathbf{R}^3 \rightarrow \mathbf{R}$ ispunjavaju:

$$\Delta g(x, y, z) = 0 \quad \text{i} \quad \Delta h(x, y, z) = 0,$$

gde je $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ - LAPLACEov operator. Za funkciju $f : \mathbf{R}^3 \rightarrow \mathbf{R}$ datu sa:

$$f(x, y, z) = g(x, y, z) + (x^2 + y^2 + z^2)h(x, y, z)$$

izračunati $\Delta \Delta f(x, y, z)$.

Rešenje. Na osnovu $\Delta g(x, y, z) = 0$ nalazimo:

$$\begin{aligned}
 \Delta f(x, y, z) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(g(x, y, z) + (x^2 + y^2 + z^2)h(x, y, z) \right) = \dots = \\
 &= 6h(x, y, z) + 4x\frac{\partial}{\partial x}h(x, y, z) + 4y\frac{\partial}{\partial y}h(x, y, z) + 4z\frac{\partial}{\partial z}h(x, y, z).
 \end{aligned}$$

Odatle, na osnovu $\Delta h(x, y, z) = 0$ nalazimo: $\Delta \Delta f(x, y, z) = 0$.

16. Neka je $f : \mathbf{R} \rightarrow \mathbf{R}$ četiri puta ferencijabilna funkcija i $r = |\vec{r}|$ – intezitet radijus vektora u \mathbf{R}^3 . Ako je $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ – LAPLACEov operator dokazati jednakosti:

$$\text{a) } \frac{\partial^2}{\partial x^2} f(r) = \frac{r^2 - x^2}{r^3} \frac{df(r)}{dr} + \frac{x^2}{r^2} \frac{d^2 f(r)}{dr^2},$$

$$\text{b) } \Delta f(r) = \frac{2}{r} \frac{df(r)}{dr} + \frac{d^2 f(r)}{dr^2},$$

$$\text{c) } \Delta^2 f(r) = \frac{4}{r} \frac{d^3 f(r)}{dr^3} + \frac{d^4 f(r)}{dr^4}.$$

Rešenje. a) Za složenu funkciju $u = f(r)$, gde je $r = \sqrt{x^2 + y^2 + z^2}$, nalazimo parcijalni izvod prvog reda:

$$\frac{\partial f(r)}{\partial x} = \frac{df}{dr} \cdot \frac{\partial f(r)}{\partial x} = f'(r) \cdot \frac{x}{r}.$$

Odatle nalazimo traženi parcijalni izvod drugog reda:

$$\begin{aligned} \frac{\partial^2 f(r)}{\partial x^2} &= \frac{\partial}{\partial x} \left(f'(r) \cdot \frac{x}{r} \right) = \frac{\partial}{\partial x} \left(f'(r) \right) \cdot \frac{x}{r} + f'(r) \cdot \frac{\partial}{\partial x} \left(\frac{x}{r} \right) \\ &= \left(f''(r) \cdot \frac{x}{r} \right) \cdot \frac{x}{r} + f'(r) \cdot \left(\frac{r - x^2/r}{r^2} \right) \\ &= \frac{r^2 - x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r). \end{aligned}$$

b) Na osnovu dokazane jednakosti pod **a)** zaključujemo:

$$\begin{aligned} \frac{\partial^2 f(r)}{\partial x^2} &= \frac{r^2 - x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r), \\ \frac{\partial^2 f(r)}{\partial y^2} &= \frac{r^2 - y^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r), \\ \frac{\partial^2 f(r)}{\partial z^2} &= \frac{r^2 - z^2}{r^3} f'(r) + \frac{z^2}{r^2} f''(r). \end{aligned}$$

Sabiranjem prethodne tri jednakosti dobijamo traženu jednakost:

$$\Delta f(r) = \frac{\partial^2 f(r)}{\partial x^2} + \frac{\partial^2 f(r)}{\partial y^2} + \frac{\partial^2 f(r)}{\partial z^2} = \frac{2}{r} \frac{df(r)}{dr} + \frac{d^2 f(r)}{dr^2}.$$

c) I Način. Na osnovu dokazane jednakosti pod **b)** zaključujemo:

$$\begin{aligned} \Delta^2 f(r) &= \Delta(\Delta f(r)) = \Delta \left(\frac{2}{r} \frac{df(r)}{dr} + \frac{d^2 f(r)}{dr^2} \right) = \Delta \left(\frac{2}{r} \frac{df(r)}{dr} \right) + \Delta \left(\frac{d^2 f(r)}{dr^2} \right) \\ &= \frac{2}{r} \frac{d}{dr} \left(\frac{2}{r} \frac{df}{dr} \right) + \frac{d^2}{dr^2} \left(\frac{2}{r} \frac{df}{dr} \right) + \Delta \left(\frac{d^2 f(r)}{dr^2} \right) \\ &= \frac{2}{r} \left(-\frac{2}{r^2} \frac{df}{dr} + \frac{2}{r} \frac{d^2 f}{dr^2} \right) + \frac{d}{dr} \left(-\frac{2}{r^2} \frac{df}{dr} + \frac{2}{r} \frac{d^2 f}{dr^2} \right) + \Delta \left(\frac{d^2 f(r)}{dr^2} \right) \\ &= \left(-\frac{4}{r^3} \frac{df}{dr} + \frac{4}{r^2} \frac{d^2 f}{dr^2} \right) + \left(\frac{4}{r^3} \frac{df}{dr} - \frac{2}{r^2} \frac{d^2 f}{dr^2} - \frac{2}{r^2} \frac{df^2}{dr^2} + \frac{2}{r} \frac{d^3 f}{dr^3} \right) + \Delta \left(\frac{d^2 f(r)}{dr^2} \right) \\ &= \frac{2}{r} \frac{d^3 f(r)}{dr^3} + \Delta \left(\frac{d^2 f(r)}{dr^2} \right) = \frac{2}{r} \frac{d^3 f(r)}{dr^3} + \left(\frac{2}{r} \frac{d}{dr} \left(\frac{d^2 f(r)}{dr^2} \right) + \frac{d^2}{dr^2} \left(\frac{d^2 f(r)}{dr^2} \right) \right) \\ &= \frac{4}{r} \frac{d^3 f(r)}{dr^3} + \frac{d^4 f(r)}{dr^4}. \end{aligned}$$

II Način. Na osnovu dokazane jednakosti pod **b)** zaključujemo:

$$\begin{aligned}
\Delta^2 f(r) &= \Delta(\Delta f(r)) = \frac{2}{r} \frac{d(\Delta f(r))}{dr} + \frac{d^2(\Delta f(r))}{dr^2} \\
&= \frac{2}{r} \frac{d}{dr} \left(\frac{2}{r} \frac{df(r)}{dr} + \frac{d^2 f(r)}{dr^2} \right) + \frac{d^2}{dr^2} \left(\frac{2}{r} \frac{df(r)}{dr} + \frac{d^2 f(r)}{dr^2} \right) \\
&= \frac{2}{r} \left(-\frac{2}{r^2} \frac{df(r)}{dr} + \frac{2}{r} \frac{d^2 f(r)}{dr^2} + \frac{d^3 f(r)}{dr^3} \right) \\
&\quad + \frac{d}{dr} \left(-\frac{2}{r^2} \frac{df(r)}{dr} + \frac{2}{r} \frac{d^2 f(r)}{dr^2} + \frac{d^3 f(r)}{dr^3} \right) \\
&= \left(-\frac{4}{r^3} \frac{df(r)}{dr} + \frac{4}{r^2} \frac{d^2 f(r)}{dr^2} + \frac{2}{r} \frac{d^3 f(r)}{dr^3} \right) \\
&\quad + \frac{d}{dr} \left(-\frac{2}{r^2} \frac{df(r)}{dr} + \frac{2}{r} \frac{d^2 f(r)}{dr^2} + \frac{d^3 f(r)}{dr^3} \right) \\
&= \left(-\frac{4}{r^3} \frac{df(r)}{dr} + \frac{4}{r^2} \frac{d^2 f(r)}{dr^2} + \frac{2}{r} \frac{d^3 f(r)}{dr^3} \right) \\
&\quad + \left(\frac{4}{r^3} \frac{df(r)}{dr} - \frac{4}{r^2} \frac{d^2 f(r)}{dr^2} + \frac{2}{r} \frac{d^3 f(r)}{dr^3} + \frac{d^4 f(r)}{dr^4} \right) \\
&= \frac{4}{r} \frac{d^3 f(r)}{dr^3} + \frac{d^4 f(r)}{dr^4}.
\end{aligned}$$

17. Pokazati da je vektorsko polje:

$$\vec{f} = (2x + y + z, x + 2y + z, x + y + 2z)$$

potencijalno i naći njegov potencijal.

Rešenje. Neposredno se proverava da za svako $(x, y, z) \in \mathbf{R}^3$ važi:

$$\operatorname{rot} \vec{f} = \vec{0} \quad \text{i} \quad \operatorname{div} \vec{f} = 6 (\neq 0).$$

Samim tim vektorsko polje \vec{f} je potencijalno. Dalje, za prethodno potencijalno polje $\vec{f}: \mathbf{R}^3 \rightarrow \mathbf{R}$, potencijal jeste svaka funkcija $g: \mathbf{R}^3 \rightarrow \mathbf{R}$ takva da važi:

$$\operatorname{grad} g = \vec{f}.$$

Prethodna jednakost dovodi do sistema:

$$(1) \quad \frac{\partial g}{\partial x} = 2x + y + z,$$

$$(2) \quad \frac{\partial g}{\partial y} = x + 2y + z,$$

$$(3) \quad \frac{\partial g}{\partial z} = x + y + 2z.$$

Iz jednačine (1) dobijamo:

$$(4) \quad g = x^2 + xy + zx + a(y, z),$$

za neku funkciju $a = a(y, z) : \mathbf{R}^2 \rightarrow \mathbf{R}$. Dalje, iz jednačina (2) i (4) dobijamo sledeću jednakost: $\frac{\partial a}{\partial y} = 2y + z$ i odatle:

$$(5) \quad a(y, z) = y^2 + yz + b(z),$$

za neku funkciju $b = b(z) : \mathbf{R} \rightarrow \mathbf{R}$. Iz jednačina (4) i (5) dobijamo:

$$(6) \quad g = x^2 + y^2 + xy + yz + zx + b(z).$$

Konačno, iz jednačina (3) i (6) dobijamo diferencijalnu jednačinu:

$$(7) \quad b'(z) - 2z = 0,$$

sa opštim rešenjem:

$$(8) \quad b(z) = z^2 + C,$$

gde je C neka konstanta. Sveukupno, funkcija potencijala je oblika:

$$g = x^2 + y^2 + z^2 + xy + yz + zx + C.$$

18. Pokazati da vrednost integrala

$$J = \int_{\widehat{AB}} (x^4 + 4xy^3)dx + (6x^2y^2 - 5y^4)dy$$

ne zavisi od krive koja spaja tačke $A(0, 0)$ i $B(\alpha, \beta)$ ($\alpha, \beta \in \mathbf{R}$). Izračunati vrednost tog integrala.

Rešenje. Za funkcije $P = P(x, y) = x^4 + 4xy^3$, $Q = Q(x, y) = 6x^2y^2 - 5y^4 : \mathbf{R}^2 \rightarrow \mathbf{R}$ ispunjen je uslov za nezavisnost krivolinijskog integrala od puta integracije:

$$\frac{\partial P}{\partial y} = 12xy^2 = \frac{\partial Q}{\partial x}.$$

Na osnovu:

$$(x^4 + 4xy^3) dx + (6x^2y^2 - 5y^4) dy = d\left(\frac{1}{5}x^5 + 2x^2y^3 - y^5\right)$$

nalazimo vrednost integrala:

$$\begin{aligned} J &= \int_{\widehat{AB}} (x^4 + 4xy^3) dx + (6x^2y^2 - 5y^4) dy \\ &= \left(\frac{1}{5}x^5 + 2x^2y^3 - y^5\right) \Big|_{(0,0)}^{(\alpha,\beta)} = \frac{1}{5}\alpha^5 + 2\alpha^2\beta^3 - \beta^5. \end{aligned}$$

19. Data su skalarna polja $f = xyz$, $g = xy + yz + zx$. 1^o. Formirati vektorska polja $\vec{a} = \text{grad} f$, $\vec{b} = \text{grad} g$ i ispitati prirodu vektorskog polja $\vec{a} \times \vec{b}$. 2^o. Izračunati $\int_C (\vec{a} \times \vec{b}) \cdot d\vec{r}$,

gde je C duž koja spaja tačke $O(0, 0, 0)$ i $B(1, 2, 3)$.

Rešenje. 1^o. Važi $\vec{a} = \text{grad} f = (yz, zx, xy)$ i $\vec{b} = \text{grad} g = (y + z, z + x, x + y)$. Na osnovu toga formiramo vektorsko polje:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ yz & zx & xy \\ y+z & z+x & x+y \end{vmatrix} = x^2(z-y)\vec{i} + y^2(x-z)\vec{j} + z^2(y-x)\vec{k}.$$

Odatle se proverava: $\text{div}(\vec{a} \times \vec{b}) = 0$ i $\text{rot}(\vec{a} \times \vec{b}) = (y^2 + z^2)\vec{i} + (z^2 + x^2)\vec{j} + (x^2 + y^2)\vec{k} \neq \vec{0}$. Samim tim polje $\vec{a} \times \vec{b}$ jeste solenoidno.

2^o. Duž OB se može prikazati u obliku $x = t$, $y = 2t$, $z = 3t$, gde je $t \in [0, 1]$. Cirkulacija vektora $\vec{a} \times \vec{b}$ po duži OB iznosi:

$$\int_C (\vec{a} \times \vec{b}) \cdot d\vec{r} = \int_C x^2(z-y)dx + y^2(x-z)dy + z^2(y-x)dz = 12 \int_0^1 t^3 dt = 3.$$

20. U prostoru \mathbf{R}^3 dato je vektorsko polje $\vec{a} = f(\vec{b} \circ \vec{r}) \cdot (\vec{b} \times \vec{r})$, gde je $f : \mathbf{R} \rightarrow \mathbf{R}$ diferencijabilna funkcija i gde je \vec{b} potencijalno polje sa potencijalom $u(x, y, z) = x^2/2 + yz - xz$. Pokazati da je vektorsko polje \vec{a} solenoidno. Ako je $f(\vec{b} \circ \vec{r}) = \vec{b} \circ \vec{r}$, naći cirkulaciju vektora \vec{a} duž pozitivno orjetisane krive koja pripada ravni Oxy i sastoji se od segmenta $[0, 1]$ sa x ose, četvrtine kruga $x^2 + y^2 = 1$ koji pripada prvom kvadrantu i segmenta $[0, 1]$ sa y ose.

Rešenje. (i) Polje \vec{b} je potencijalno sa potencijalom $u = u(x, y, z) = x^2/2 + yz - xz$. Odatle $\vec{b} = \text{grad} u = (x - z, z, y - x)$. Direktno nalazimo skalarni proizvod:

$$\vec{b} \circ \vec{r} = (x - z, z, y - x) \circ (x, y, z) = x^2 - 2xz + 2yz$$

i vektorski proizvod:

$$\begin{aligned} \vec{b} \times \vec{r} &= (x - z, z, y - x) \times (x, y, z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x - z & z & y - x \\ x & y & z \end{vmatrix} \\ &= (z^2 - y^2 + xy, xy - x^2 - xz + z^2, xy - yz - xz). \end{aligned}$$

Na osnovu prethodnog dobijamo traženo vektorsko polje:

$$\begin{aligned} \vec{a} &= f(\vec{b} \circ \vec{r}) \cdot (\vec{b} \times \vec{r}) \\ &= f(x^2 - 2xz + 2yz) \cdot (z^2 - y^2 + xy, xy - x^2 - xz + z^2, xy - yz - xz), \end{aligned}$$

gde je $f : \mathbf{R} \rightarrow \mathbf{R}$ proizvoljna diferencijabilna funkcija.

(ii) Ako uvedemo pomoćnu funkciju $u = x^2 - 2xz + 2yz$, tada važi:

$$\begin{aligned}
\operatorname{div} \vec{a} &= \left(f'(u) \underbrace{(2x - 2z)}_{u'_x} \cdot (z^2 - y^2 + xy) + y \cdot f(u) \right) \\
&+ \left(f'(u) \underbrace{(2z)}_{u'_y} \cdot (xy - x^2 - xz + z^2) + x \cdot f(u) \right) \\
&+ \left(f'(u) \underbrace{(2y - 2x)}_{u'_z} \cdot (xy - yz - xz) + (-y - x) \cdot f(u) \right) \\
&= f'(u) \left((2x - 2z)(z^2 - y^2 + xy) + (2z)(xy - x^2 - xz + z^2) + (2y - 2x)(xy - yz - xz) \right) \\
&+ (y + x - y - x) \cdot f(u) \\
&= f'(u) \left((2xz^2 - 2xy^2 + 2x^2y - 2z^3 + 2zy^2 - 2xyz) + (2xyz - 2x^2z - 2xz^2 + 2z^3) \right) \\
&+ (2xy^2 - 2y^2z - 2xyz - 2x^2y + 2xyz + 2x^2z) = 0
\end{aligned}$$

Na sličan način može se pokazati da je $\operatorname{rot} \vec{a} \neq \vec{0}$, pri tom za dokaz prethodnog tvrđenja dovoljno je dokazati da bar po jednoj koordinati funkcija rotora nije jednaka nuli. Odatle sleduje zaključak da je vektorsko polje \vec{a} solenoidno. Napomenimo da do prethodnog rezultata možemo doći i primenom simboličkog računa.

(iii) Specijalno za $f(t) = t$ imamo vektorsko polje:

$$\vec{a} = (x^2 - 2xz + 2yz) \cdot (z^2 - y^2 + xy, xy - x^2 - xz + z^2, xy - yz - xz)$$

koje u ravni $z = 0$ određuje vektorsko polje: $\vec{a} = (x^2) \cdot (xy - y^2, xy - x^2, xy)$. Odatle, cirkulacija je data integralom:

$$I = \int_L \vec{a} \circ d\vec{r} = \int_L (x^3y - x^2y^2, x^3y - x^4, x^3y) \circ (dx, dy, dz),$$

odnosno:

$$I = \int_L (x^3y - x^2y^2)dx + (x^3y - x^4)dy.$$

Označimo sa D oblast koju obuhvata pozitivno orjetisana kontura L koja pripada ravni Oxy i sastoji se od segmenta $[0, 1]$ sa x ose, četvrtine kruga $x^2 + y^2 = 1$ koji pripada prvom kvadrantu i segmentu $[0, 1]$ sa y ose. Funkcije:

$$P(x, y) = x^3 - x^2y^2, \quad Q(x, y) = x^3y - x^4 : D \xrightarrow{\mathbf{R}^2} \mathbf{R}$$

ispunjavaju uslove za primenu GREENOVE formule. Samim tim:

$$I = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = 5 \iint_D (x^2y - x^3) dx dy.$$

Prelaskom na polarne koordinate: $\{x = \rho \sin \varphi, y = \rho \cos \varphi\}$ ($|J| = \rho$), vrednost cirku-

lacije duž pozitivno orjetisane krive L je data integralom:

$$\begin{aligned}
 I &= 5 \int_0^{\pi/2} \int_0^1 (\rho^3 \cos^2 \varphi \sin \varphi - \rho^3 \cos^3 \varphi) \rho \, d\varphi \, d\rho \\
 &= 5 \int_0^{\pi/2} (\cos^2 \varphi \sin \varphi - \cos^3 \varphi \sin^0 \varphi) \, d\varphi \int_0^1 \rho^4 \, d\rho \\
 &= 5 \left(\frac{1}{2} B\left(\frac{2+1}{2}, \frac{1+1}{2}\right) - \frac{1}{2} B\left(\frac{3+1}{2}, \frac{0+1}{2}\right) \right) \frac{1}{5} = -\frac{1}{3}.
 \end{aligned}$$

21. Izračunati fluks vektora $\vec{f} = (x, -y^2, x^2 + z^2 - 1)$ iznutra površi elipsoida:

$$\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1.$$

Rešenje. Fluks vektora kroz površ iznutra te površi (koji uvire u površ) računa se po formuli:

$$\begin{aligned}
 \Phi &= \iint_{S^-} (x, -y^2, x^2 + z^2 - 1) \circ (dydz, dzdx, dxdy) \\
 &= \iint x \, dydz - y^2 \, dzdx + (x^2 + z^2 - 1) \, dxdy,
 \end{aligned}$$

gde je S elipsoid $\frac{x^2}{3^2} + \frac{y^2}{2^2} + \frac{z^2}{1^2} = 1$. Kako su ispunjeni uslovi za primenu teoreme OSTROGRADSKOG, tada važi:

$$\Phi = - \iiint_{S^+} x \, dydz - y^2 \, dzdx + (x^2 + z^2 - 1) \, dxdy = - \iiint_V (1 - 2y + 2z) \, dxdydz,$$

gde je V unutrašnjost elipsoida $\frac{x^2}{3^2} + \frac{y^2}{2^2} + \frac{z^2}{1^2} \leq 1$. Uvođenjem uopšetnih sfernih koordinata: $x = 3\rho \cos \theta \cos \varphi$, $y = 2\rho \cos \theta \sin \varphi$, $z = \rho \sin \theta$ (sa vrednošću $|J| = 6\rho^2 \cos \theta$) oblast V_1 : $0 \leq \rho \leq 1 \wedge -\pi/2 \leq \theta \leq \pi/2 \wedge 0 \leq \varphi \leq 2\pi$ se skoro svuda jednoznačno preslikava na oblast V . Na osnovu toga:

$$\begin{aligned}
 \Phi &= - \iiint_{V_1} \left(1 - \underbrace{2(2\rho \cos \theta \sin \varphi)}_y + \underbrace{2(\rho \sin \theta)}_z \right) \underbrace{6\rho^2 \cos \theta}_{|J|} \, d\rho d\theta d\varphi \\
 &= -6 \int_0^1 \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} (\rho^2 \cos \theta - 4\rho^3 \cos^2 \theta \sin \varphi + 2\rho^3 \cos \theta \sin \theta) \, d\rho d\theta d\varphi \\
 &= -6 \int_0^1 \int_{-\pi/2}^{\pi/2} \left(\frac{1}{3} \cos \theta - 4\frac{1}{4} \cos^2 \theta \sin \varphi + 2\frac{1}{4} \cos \theta \sin \theta \right) d\theta d\varphi \\
 &= -6 \int_{-\pi/2}^{\pi/2} \left(\frac{2\pi}{3} \cos \theta - \cos^2 \theta \int_0^{2\pi} \sin \varphi \, d\varphi + \pi \cos \theta \sin \theta \right) d\theta \\
 &= -6 \int_{-\pi/2}^{\pi/2} \frac{2\pi}{3} \cos \theta \, d\theta = -8\pi.
 \end{aligned}$$

22. Neka je $f: \mathbf{R} \rightarrow \mathbf{R}$ bar dva puta neprekidno diferencijabilna funkcija. Dokazati:

a) $\Delta f(r) = \frac{2}{r}f'(r) + f''(r)$, gde je $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ – LAPLACEov operator

i gde je $r = |\vec{r}|$ – intezitet radijus vektora u \mathbf{R}^3 .

b) Za vektorsko polje: $\vec{a} = r^n \text{grad}f(r)$ ($n \geq 2$) odrediti funkciju $f(r)$ tako da je polje \vec{a} LAPLACEovo i da pri tom važi: $f(1) = \alpha$ i $f'(1) = \beta$ za $\alpha, \beta \in \mathbf{R}$.

Rešenje. I Način. Zadatak se može uraditi i direktnim računom po koordinatama.

a) Važi:

$$\begin{aligned} \Delta f(r) &= \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) f(r) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial z} \right) \\ &= \frac{\partial}{\partial x} \left(f' \cdot \frac{x}{r} \right) + \frac{\partial}{\partial y} \left(f' \cdot \frac{y}{r} \right) + \frac{\partial}{\partial z} \left(f' \cdot \frac{z}{r} \right) = \left(\frac{\partial f'}{\partial x} \frac{x}{r} + f' \frac{y^2 + z^2}{r^3} \right) + \left(\frac{\partial f'}{\partial y} \frac{y}{r} \right. \\ &+ \left. f' \frac{x^2 + z^2}{r^3} \right) + \left(\frac{\partial f'}{\partial z} \frac{z}{r} + f' \frac{x^2 + y^2}{r^3} \right) = f' \frac{2r^2}{r^3} + f'' \left(\frac{x}{r} \right)^2 + f'' \left(\frac{y}{r} \right)^2 + f'' \left(\frac{z}{r} \right)^2 \\ &= \frac{2}{r} f'(r) + f''(r). \end{aligned}$$

b) Vektorsko polje \vec{a} dato je sa vektorom:

$$\vec{a} = r^n \cdot \text{grad}(f(r)) = \left(r^n \frac{\partial f}{\partial x} \right) \vec{i} + \left(r^n \frac{\partial f}{\partial y} \right) \vec{j} + \left(r^n \frac{\partial f}{\partial z} \right) \vec{k}.$$

Rotor vektorskog polja \vec{a} koordinatno računat iznosi:

$$\begin{aligned} \text{rot } \vec{a} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(r^n \frac{\partial f}{\partial x} \right) & \left(r^n \frac{\partial f}{\partial y} \right) & \left(r^n \frac{\partial f}{\partial z} \right) \end{vmatrix} = \left(\frac{\partial}{\partial y} \left(r^n \cdot \frac{\partial f}{\partial z} \right) - \frac{\partial}{\partial z} \left(r^n \cdot \frac{\partial f}{\partial y} \right) \right) \vec{i} \\ &+ \left(\frac{\partial}{\partial z} \left(r^n \cdot \frac{\partial f}{\partial x} \right) - \frac{\partial}{\partial x} \left(r^n \cdot \frac{\partial f}{\partial z} \right) \right) \vec{j} + \left(\frac{\partial}{\partial x} \left(r^n \cdot \frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left(r^n \cdot \frac{\partial f}{\partial x} \right) \right) \vec{k} \\ &= \left(nr^{n-1} \frac{\partial r}{\partial y} \cdot f'(r) \frac{\partial r}{\partial z} + r^n \frac{\partial^2 f}{\partial y \partial z} - nr^{n-1} \frac{\partial r}{\partial z} \cdot f'(r) \frac{\partial r}{\partial y} - r^n \frac{\partial^2 f}{\partial z \partial y} \right) \vec{i} \\ &+ \left(nr^{n-1} \frac{\partial r}{\partial z} \cdot f'(r) \frac{\partial r}{\partial x} + r^n \frac{\partial^2 f}{\partial z \partial x} - nr^{n-1} \frac{\partial r}{\partial x} \cdot f'(r) \frac{\partial r}{\partial z} - r^n \frac{\partial^2 f}{\partial x \partial z} \right) \vec{j} \\ &+ \left(nr^{n-1} \frac{\partial r}{\partial x} \cdot f'(r) \frac{\partial r}{\partial y} + r^n \frac{\partial^2 f}{\partial x \partial y} - nr^{n-1} \frac{\partial r}{\partial y} \cdot f'(r) \frac{\partial r}{\partial x} - r^n \frac{\partial^2 f}{\partial y \partial x} \right) \vec{k} = \vec{0}. \end{aligned}$$

Divergencija vektorskog polja \vec{a} koordinatno računata iznosi:

$$\begin{aligned} \text{div } \vec{a} &= \frac{\partial}{\partial x} \left(r^n \cdot \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left(r^n \cdot \frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial z} \left(r^n \cdot \frac{\partial f}{\partial z} \right) \\ &= nr^{n-1} \frac{\partial r}{\partial x} \cdot \frac{\partial f}{\partial x} + r^n \frac{\partial^2 f}{\partial x^2} + nr^{n-1} \frac{\partial r}{\partial y} \cdot \frac{\partial f}{\partial y} + r^n \frac{\partial^2 f}{\partial y^2} + nr^{n-1} \frac{\partial r}{\partial z} \cdot \frac{\partial f}{\partial z} + r^n \frac{\partial^2 f}{\partial z^2} \\ &= nr^{n-1} \frac{\partial f}{\partial r} \left(\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 + \left(\frac{\partial r}{\partial z} \right)^2 \right) + r^n \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) \\ &= nr^{n-1} \cdot f'(r) + r^n \cdot \Delta f(r) \stackrel{\text{a)}}{=} nr^{n-1} \cdot f'(r) + r^n \cdot \left(\frac{2}{r} f'(r) + f''(r) \right) \\ &= (n+2)r^{n-1} \cdot f'(r) + r^n \cdot f''(r). \end{aligned}$$

Ispitajmo kad je vektorsko polje $\vec{a} = r^n \text{grad } f(r)$ LAPLACEovo, tj. kad važi:

$$\text{rot } \vec{a} = \vec{0} \wedge \text{div } \vec{a} = 0 \iff (n+2)r^{n-1} \cdot f'(r) + r^n \cdot f''(r) = 0 \quad (*)$$

Trivijalno rešenje diferencijalne jednačine (*) jeste konstantna funkcija $f(r) = c$, gde je c neka konstanta. Primito da ako je ispunjeno $\alpha \neq c \vee \beta \neq 0$, tada trivijalno rešenje ne zadovoljava postavljene početne uslove. Dalje, pod pretpostavkom da f nije konstantna funkcija i da $r \neq 0$, važi:

$$\begin{aligned} \text{div } \vec{a} = 0 &\iff (n+2)r^{n-1} \cdot f'(r) + r^n \cdot f''(r) = 0 \iff \frac{f''(r)}{f'(r)} = -\frac{(n+2)}{r} \\ &\iff f'(r) = \frac{c_1}{r^{n+2}} \iff f(r) = -\frac{c_1}{(n+1)r^{n+1}} + c_2, \end{aligned}$$

za neke konstante $c_1, c_2 \in \mathbf{R}$. Iz početnih uslova dobijamo:

$$f(1) = -\frac{c_1}{n+1} + c_2 = \alpha \quad \wedge \quad f'(1) = c_1 = \beta.$$

Odatle nalazimo: $c_1 = \beta$ i $c_2 = \alpha + \frac{\beta}{n+1}$. Samim tim, tražena funkcija f data je u obliku:

$$f(r) = -\frac{\beta}{(n+1)r^{n+1}} + \alpha + \frac{\beta}{n+1} \quad (r \neq 0).$$

II Način. a) Primenenom simboličkog računa nad vektorima važi:

$$\begin{aligned} \Delta(f(r)) &= \text{div}(\text{grad}(f(r))) = \text{div}(f'(r) \cdot \text{grad}(r)) = \text{div}(f'(r) \cdot \vec{r}_0) \\ &= f'(r) \cdot \text{div}(\vec{r}_0) + \vec{r}_0 \circ \text{grad}(f'(r)) = f'(r) \cdot \frac{2}{r} + \vec{r}_0 \circ (f''(r) \cdot \underbrace{\text{grad}(r)}_{=\vec{r}_0}) \\ &= \frac{2}{r}f'(r) + f''(r). \end{aligned}$$

b) Primenom simboličkog računa rotor vektorskog polja $\vec{a} = r^n \text{grad } f(r)$ iznosi:

$$\text{rot } \vec{a} = \text{rot}(r^n \text{grad } f(r)) = r^n \underbrace{\text{rot grad } f(r)}_{=\vec{0}} - \text{grad } f(r) \times \text{grad}(r^n) = -f'(r)\vec{r}_0 \times nr^{n-1}\vec{r}_0 = \vec{0}.$$

Primenom simboličkog računa divergencija vektorskog polja $\vec{a} = r^n \text{grad } f(r)$ iznosi:

$$\begin{aligned} \text{div } \vec{a} &= \text{div}(r^n \text{grad } f(r)) = r^n \underbrace{\text{div grad } f(r)}_{=\Delta} + \text{grad}(r^n) \circ \text{grad } f(r) \\ &= r^n \cdot \Delta f(r) + (nr^{n-1} \cdot \vec{r}_0) \circ (f'(r) \cdot \vec{r}_0) \\ &= \underset{\text{a)}}{nr^{n-1}} \cdot f'(r) + r^n \cdot \left(\frac{2}{r}f'(r) + f''(r)\right) \\ &= (n+2)r^{n-1} \cdot f'(r) + r^n \cdot f''(r). \end{aligned}$$

Dalje, nepoznata funkcija $f(r)$ određuje se na isti način kao u prethodnom rešenju.

23. U prostoru \mathbf{R}^3 dato je vektorsko polje $\vec{a}(\vec{r}) = r^n \cdot \vec{r}$, gde je \vec{r} vektor položaja, $r = |\vec{r}|$ i $n \in \mathbf{N}$. Dokazati da je \vec{a} potencijalno polje i odrediti potencijal tog polja. Odrediti fluks polja \vec{a} kroz spoljašnju površ polusfere $x^2 + y^2 + z^2 = R^2$, $z > 0$ ($R > 0$).

Rešenje. Primenom simboličkog računa nad vektorskim poljem $\vec{a}(\vec{r}) = r^n \cdot \vec{r}$ dobijamo:

$$\text{rot}(r^n \cdot \vec{r}) = r^n \cdot \text{rot}(\vec{r}) - \vec{r} \times \text{grad}(r^n) = -\vec{r} \times \text{grad}(r^n) = -\vec{r} \times (nr^{n-2} \cdot \vec{r}) = \vec{0}.$$

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$$\text{div}(r^n \cdot \vec{r}) = r^n \cdot \text{div}(\vec{r}) + \vec{r} \circ \text{grad}(r^n) = r^n 3 + \vec{r} \circ (nr^{n-2} \cdot \vec{r}) = (n+3)r^n \neq 0.$$

Samim tim polje $\vec{a} = \vec{a}(\vec{r})$ je potencijalno. S obzirom da je $\text{grad}(r^n) = nr^{n-2} \cdot \vec{r}$, zaključujemo da je potencijal polja \vec{a} dat funkcijom:

$$g_{\vec{a}}(\vec{r}) = \frac{r^{n+2}}{n+2}.$$

Na polusferi S je $d\vec{\sigma} = \vec{r}_0 d\sigma$, pa je fluks vektora \vec{a} :

$$\begin{aligned} \Phi_{\vec{a}} &= \int_S \vec{a} \circ d\vec{\sigma} = \iint_S (r^n \vec{r}) \circ \vec{r}_0 d\sigma = \iint_S r^{n+1} d\sigma = \iint_S R^{n+1} d\sigma \\ &= R^{n+1} \iint_S d\sigma = R^{n+1} \cdot P(S) = R^{n+1} \cdot (2\pi R^2) = 2\pi R^{n+3}. \end{aligned}$$